Introduction to Graphs

Introductions and Early Motivation

Course Website: https://courses.math.rochester.edu/current/248

Questions about Student Backgrounds and Interest: (answers to be submitted to me)

- What are your names and what's a fun fact about you?
- Which of these topics have you studied in the past/feel comfortable with the basics of?

Math	Linear algebra, Set Theory, Combinatorics, Probability, Proof Writing
CS	Basic programming, algorithms or data structures

- What are your main topics of interest? What subjects motivate your interest most/what would you like to get out of this class?

Idea:

Graphs provide a mathematical way to encode the idea of *adjacency*

Example: (Koenigsberg Bridge Problem)

Can one cross all seven bridges in Koenigsberg without crossing any of them twice and end up back where they started?

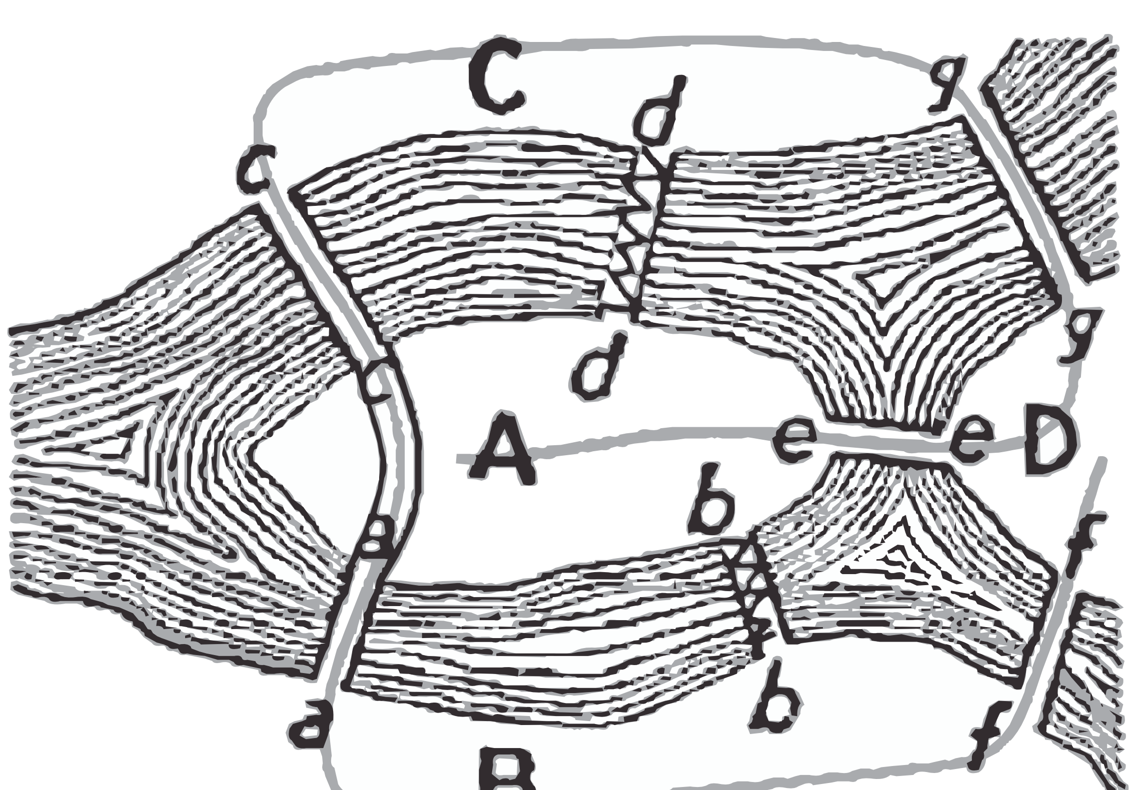
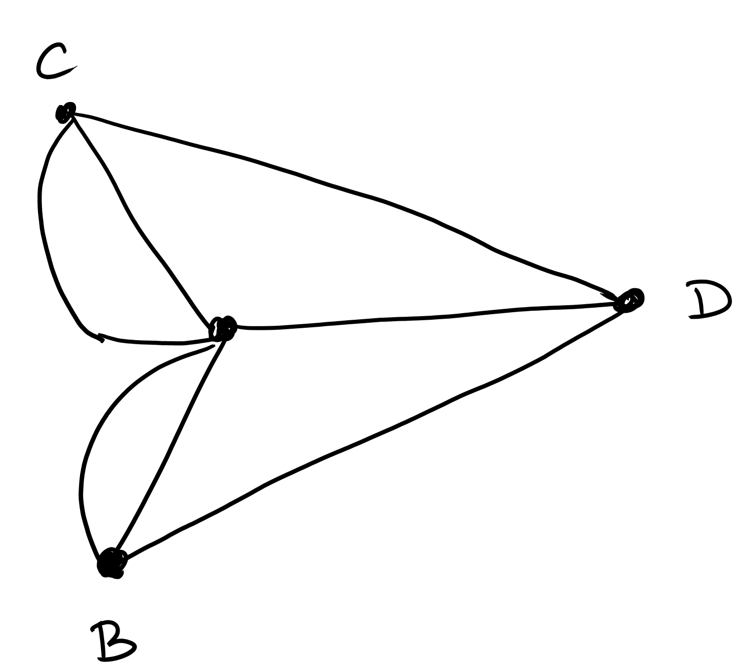
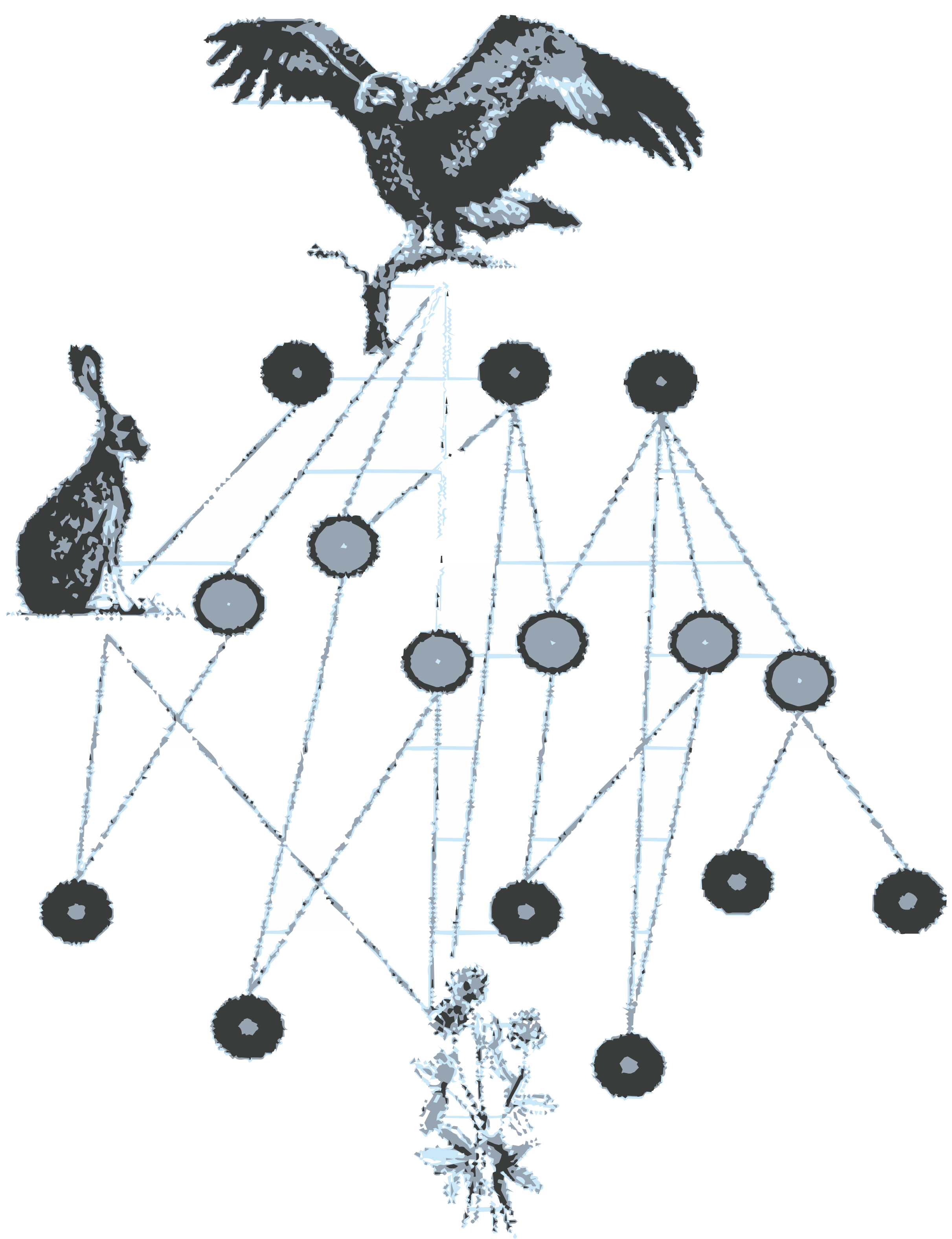


FIGURE 98. *Geographic Map: The Königsberg Bridges.*



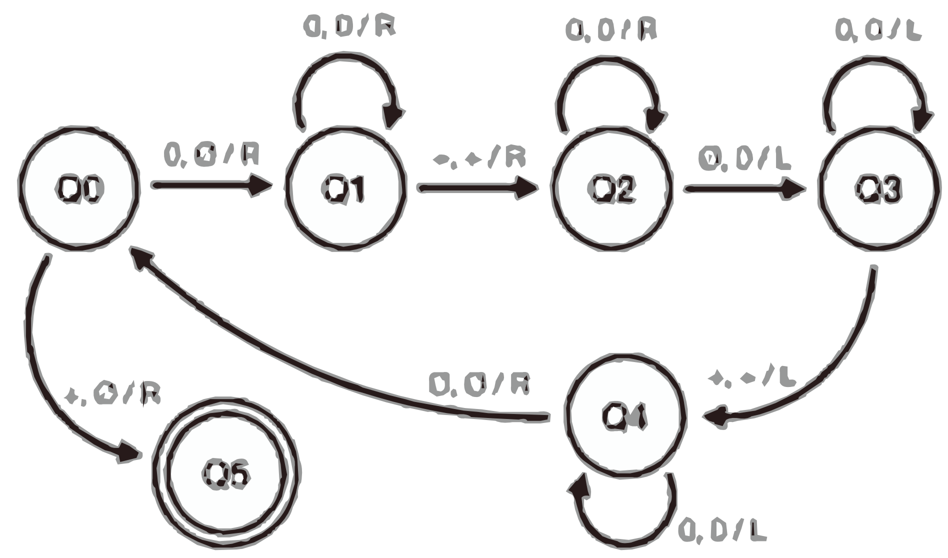
Example: (Biological, Ecological, Chemical Networks)

- How robust is a food web to the extinction of a single species?
- Which stages of a complex chemical process are rate-limiting?
- Given a large complicated system with many parts (many species, many chemical compounds, many proteins, etc.) how can we measure the complexity of the structure in a rigorous way?



Example: (Computational and Algorithmic Questions)

- How can we mathematize the idea of computation?



- How can we encode the intuitive idea of a network in a way computers can effectively process?

Definition:

A *graph* G is a triple consisting of

a vertex set V or $V(G)$

an edge set E or $E(G)$

a relation from E to V associating each edge to either one or two vertices (called *endpoints*)

We will often write $G = (V, E)$ for convenient shorthand

What is a relation? $R \subset E \times V$

Does it make sense to ask if R is an equivalence relation?

No. There are usually $R \subset S \times S$ that is

1) Reflexive $(x,x) \in R$

2) Symmetric $(x,y) \in R \Rightarrow (y,x) \in R$

3) Transitive $(x,y) \in R,\ (y,z) \in R \Rightarrow (x,z) \in R.$

The EQUALITY relation on S is the "diagonal" in $S \times S$

$R = \{(x,x) : x \in S\}$

Note:

One may wish to exclude the null graph with $V = \emptyset$ from consideration.

Review as Necessary:

- Basic set theory
  - equality of sets
  - set notation
  - empty set
  - subsets
  - cardinality
  - union, intersection, complement
  - disjointness of sets, partitions of sets
  - Cartesian product and tuples
  - relations, equivalence relations, equivalence classes
  - basic modular arithmetic as an example of the previous

- Proof theory
  - What is a proposition
  - conditional statements
  - contrapositives of statements
  - logical quantifiers, basic structure of proofs of them, negation of quantifiers
  - Direct proof, contrapositive proof, proof by contradiction
  - Induction
  - Recurrence relations
- Functions
  - Functions as mappings from domain to codomain
  - composition of functions
  - injectivity and surjectivity
  - bijections and inverse functions
  - growth rates of basic functions (bounded functions, logarithms, polynomials, exponentials, factorials, etc.)
- Combinatorics
  - summation notation over finite sets
  - permutations of finite sets
  - binomial coefficients, n choose k, the Binomial Theorem
  - Combinatorial proofs
  - Pigeonhole Principle

I have notes on induction, contrapositives and stuff like that in a note titled "Appendix 1.1 .no pp". It has some nice stuff about the difference between pf by contradiction and proving the contrapositive.

To show $P \Rightarrow Q$, assume $P \wedge \neg Q$

Prove $\neg Q \Rightarrow \neg P$

But many of the example problems involve knowing what a graph is first!

Graph Theory!

Definition:

A *loop* is an edge where both endpoints are the same

Definition:

Multiple edges are a collection of at least two edges all having the same endpoints as one another.

Is the bridges of Königsberg problem a graph or multigraph?

A Basic Classification:

	May Not Have Loops	**May Have Loops**
May Not Have Multiple Edges	Simple Graph	
May have Multiple Edges	Multigraph	Pseudograph (or sometimes Multigraph)

In what follows, we will in general assume that the term "graph" refers to a simple graph unless otherwise specified.

Definition:

A graph G is finite if both V and E are finite sets.

Definition:

If two vertices $u, v \in V$ are the endpoints of an edge in $e \in E$, we say that they are *adjacent*, that they are *neighbors*, that e *connects* u and v, and we write $u \leftrightarrow v$

Note:

For a simple graph, we can identify edges with their two endpoints, so we will often refer to "the edge uv" to denote the edge connecting these two vertices

Note:

We can think of a simple graph as a pictorial representation of a symmetric (but not reflexive!) relation on V

We originally defined a graph as $(V, E, R \subset E \times V)$. Will this allow loops and multiple edges? YES!

Motivating Question:

Does every group of six people contain a subset consisting of three people such that either

none of those people know either of the other two

each of those people knows both of the other two

?

(this is an example of a *clique finding problem*)

Let V be a set of 6 people

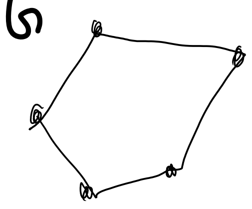
Define a graph with edges connecting pairs of people who know each other

We could also define a graph with edges connecting pairs of people who don't know each other

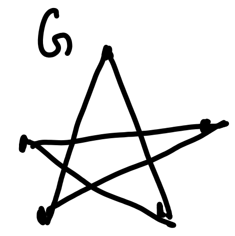
This will be on your HW

Complements

G



$\overline{G}$



pentagon
pentagram

Definition:

Let $G = (V, E)$ be a simple graph. The *complement* of G is the graph $\overline{G}$ on the same vertex set V with edge set $\overline{E}$ such that $uv \in \overline{E}$ if and only if $uv \notin E$

Definition:

A *clique* is set of vertices which are pairwise adjacent.

Definition:

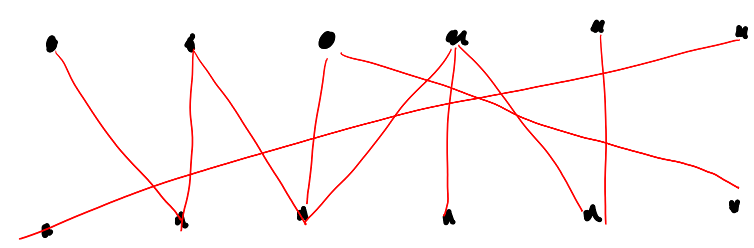
An *independent set* is a set of vertices which are pairwise nonadjacent

Motivating Question:

Suppose that you are attempting a complicated group project with several parts. Each person in your group is to do one part. Knowing that each person may be better at some tasks than others, how can you assign tasks to people?

(this is an example of a *matching problem*)

We could draw a graph with vertices consisting of tasks and people, connecting each person to all of the tasks they can do well.

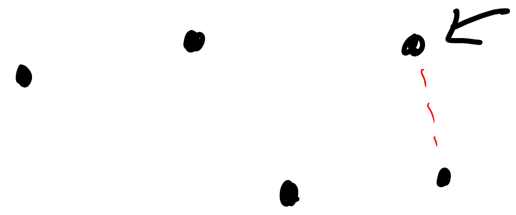


(Bipartite graph

Definition:

A graph $G = (V, E)$ is *bipartite* if $V = V_1 \cup V_2$ is the union of two disjoint independent sets.

Instead of Matthew's Sudoku puzzle, lets do assigning committees to timeslots:



← committees.

Committees are adjacent if They share have a common member.

Color the committees such that no 2 adjacent vertices have the same #.

Motivating Question:

How many colors do we need in order to draw a map of the world so that no adjacent countries have the same color?

Definition:

The *chromatic number* of a graph G is $\chi(G)$, given by the minimum number of distinct colors needed to label vertices so that all adjacent vertices receive different colors

Definition:

A graph G is *k-partite* if $V(G)$ can be written as the union of k disjoint (possibly empty) independent sets - called *partite sets*. (possibly empty disjoint union)

Proposition:

A graph G is k-partite if and only if $\chi(G) \le k$

vertices given the same color must form an indep set.

key is that any indep. set can be broken up into more (possibly empty) indep sets

Motivating Question: (for paths)

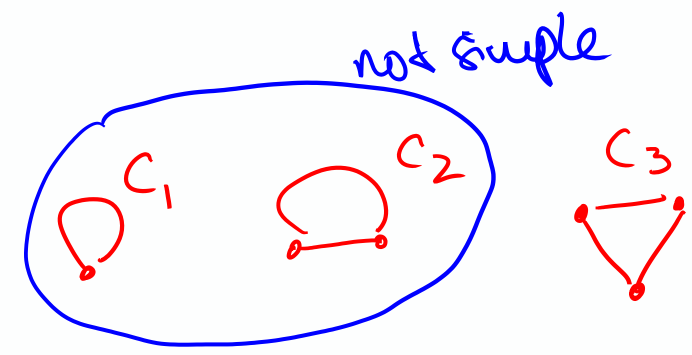
What is the fastest route for me to travel to my home?

Let the vertex set represent road intersections and the edges the roads connecting them
Label each edge by distance or travel time

Definition:

→ implies vertices cannot be repeated (no loops)

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.



Definition:

A *cycle* is a graph with an equal number of vertices and edges which can be drawn in a circle with consecutive edges in the circle connected.

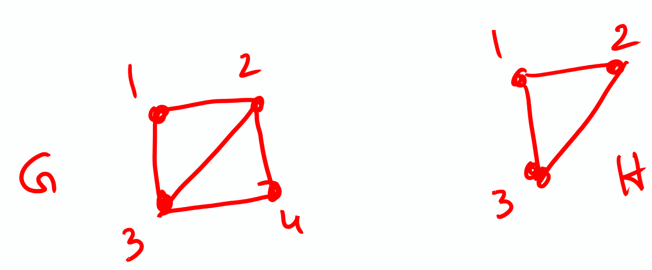
Prop: A cycle is a graph where removing any one edge turns it into a path.

Definition:

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that any edge in $E(H)$ has the same endpoints in H that it does in G

Take $xy \in H$. $xy \in E(H)$ iff $xy \in E(G)$

We say G *contains* H or *contains a copy of* H



Definition:

A graph G is *connected* iff each pair of vertices in G belongs to a path contained in G
Otherwise, G is *disconnected.*

Matrix Representation of a Graph

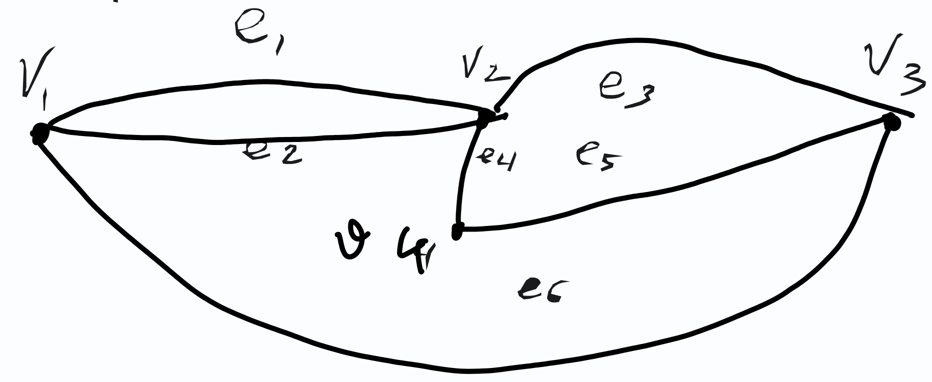
Definition:

Let G be a graph without loops. Suppose that we fix the order of the vertices in $V(G)$ as $v_1, v_2, \dots, v_n$
The *adjacency matrix* of G is the n by n matrix $A(G)$ with entries $a_{i,j}$ given by the number of edges in G with endpoints $\{v_i, v_j\}$

Allows multiple edges but no loops. why? It introduces an unnecessary confusion about whether to count loops twice or not. Both conventions appear, but typically loops add two to the degree and are counted twice in undirected graphs

Now, let us also fix the order of the edges in $E(G)$ as $e_1, \dots, e_m$.
The *incidence matrix* of G is the n by m matrix $M(G)$ with entries $m_{i,j}$ equal to 1 if v_i is an endpoint of edge e_j, and equal to 0 otherwise.

Ex:



What are the adjacency matrix and incidence matrix of this graph?

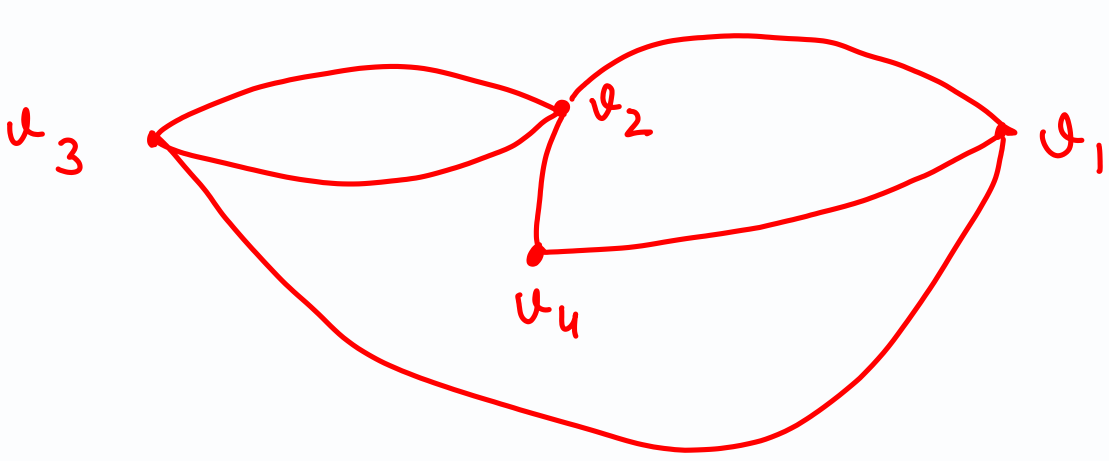
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

rowsum?

why is $a_{ij} = a_{ji}$

SKIP

$$\sigma = 3214 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$\downarrow$ permute rows

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$\downarrow$ permute columns

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Question:

Suppose we consider the square of the adjacency matrix $A^2(G) = A(G) \times A(G)$ for the above graph.

How can we interpret the entries of this matrix?

leave as open question if they don't figure it out.

$$A^2(G) = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 1 & 2 & 3 \\ 1 & 6 & 3 & 3 \\ 2 & 3 & 3 & 1 \\ 3 & 3 & 1 & 2 \end{pmatrix}$$

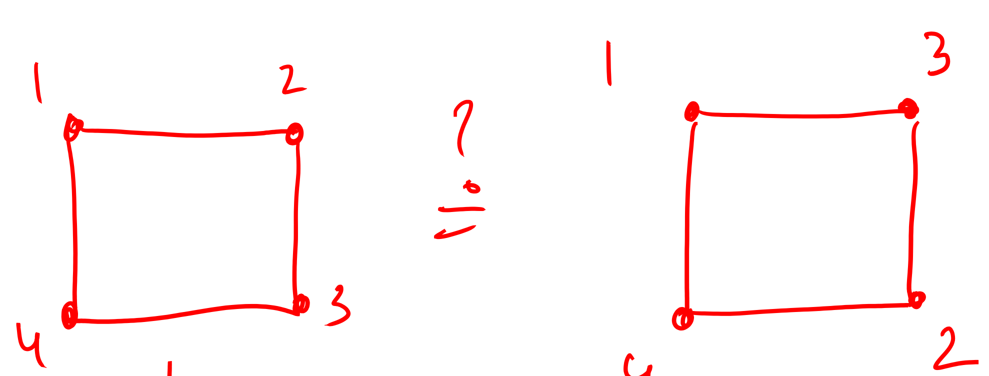
Question: What happens to the adjacency matrix/incidence matrix if we list the vertices/edges in a different order?

Question:

Construct a graph which has the following adjacency matrix

$$\begin{pmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

What if we don't really care about the specific labels we've given to a vertices of a graph? We'll often care more about "structural properties" of a graph that would be the same no matter what we called the vertices.



$(\{1,2,3,4\}, \{12,23,34\})$ $\qquad$ $(\{1\,2\,3\,4\}, \{13, 32, 24\})$

They are the "same". So want to make this precise.

Definition:

Let G and H be simple graphs. An *isomorphism* or *graph isomorphism* is a bijection $f: V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$

If there exists a bijection between G and H, we write $G \cong H$

Take C_4 above

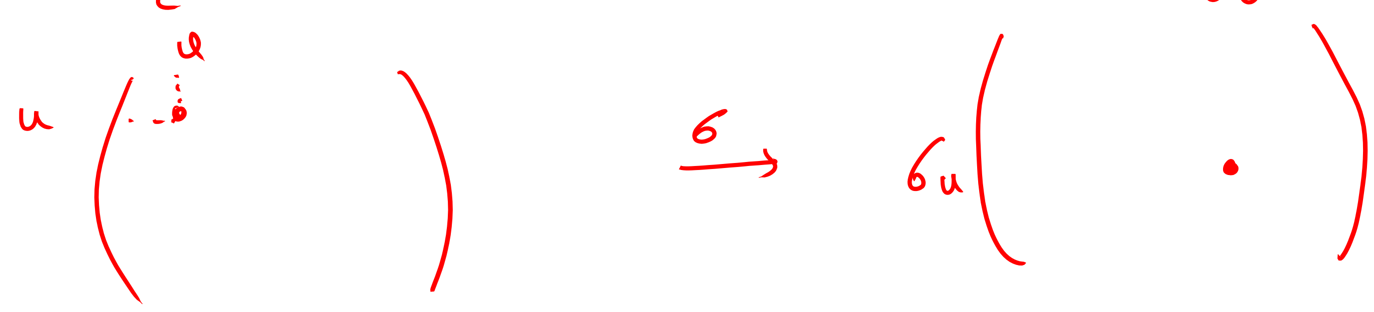
$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$\sigma : V \longrightarrow V$ (permutation)

$uv \longrightarrow \sigma u \, \sigma v$ (induces a map on edges)

If $E(H) = \{ \sigma u \, \sigma v : uv \in E(G) \}$ we're done.

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

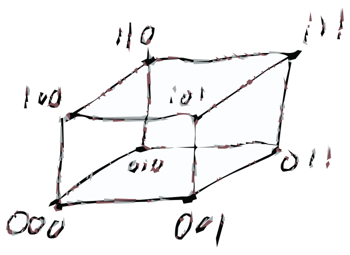


$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{permute columns}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

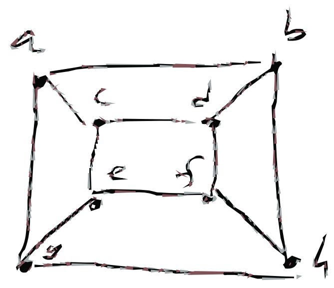
$\Rightarrow \sigma$ is an isomorphism from G to H!

G is "isomorphic" to H

Ex G_1

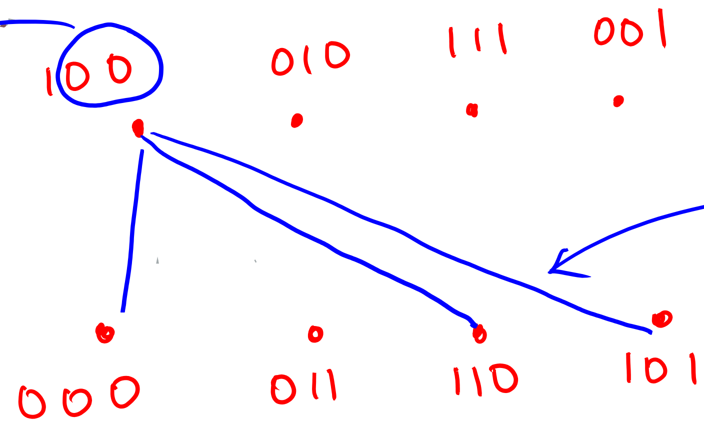


G_2



Are these isomorphic?

choose 2 words to flip



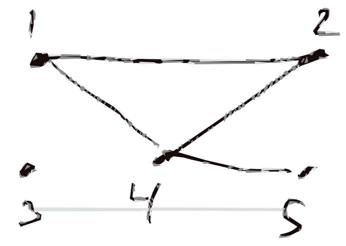
degree 3.

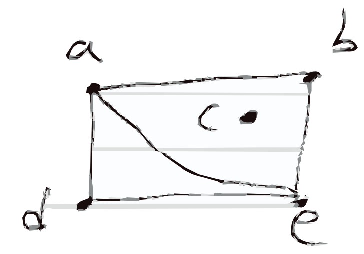
This is a good example. Shows how this hypercube is bipartite. ☆ Good problem for hypercube.

Can you generalize this.

Pairs of graphs are either isomorphic or not.

Ex: Are these isomorphic?





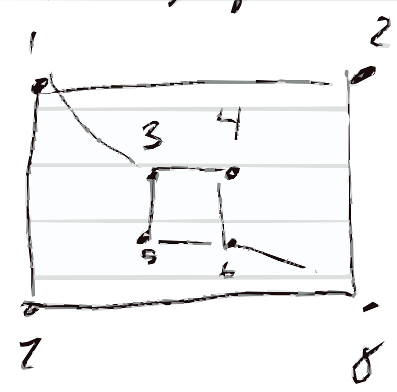
← edges have wrong degree

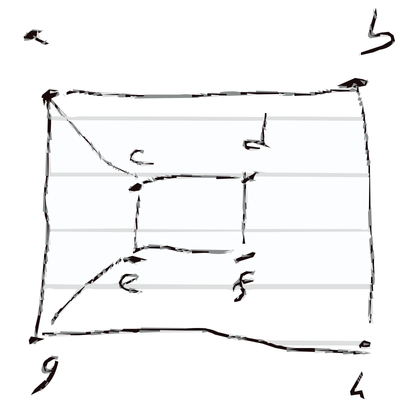
★ Show vertex degree are a graph invariant.

Idea: A graph invariant is a quantity that can be calculated for a graph s.t. it is always the same if 2 graphs are isomorphic

→ i.e. # of edges, # of vertices, connectedness, etc., girth (minimal cycle length)

Ex: Are these graphs isomorphic?





Recall: Graphs could be thought of as $(V, E, R \subset E \times V)$

relation

Simple graphs " " $(V, R \subset V \times V)$

A relation on V, is a subset of $V \times V$.

Let $M = \{$set of all graphs$\}$. The ISOMORPHISM RELATION is a subset of $M \times M$. $(G, H) \in ISO$ if $G \cong H$.

Prop. Isomorphism is an equivalence relation. on the set of all graphs

What is a relation? Sets S & T $R \subset S \times T$ is a relation.

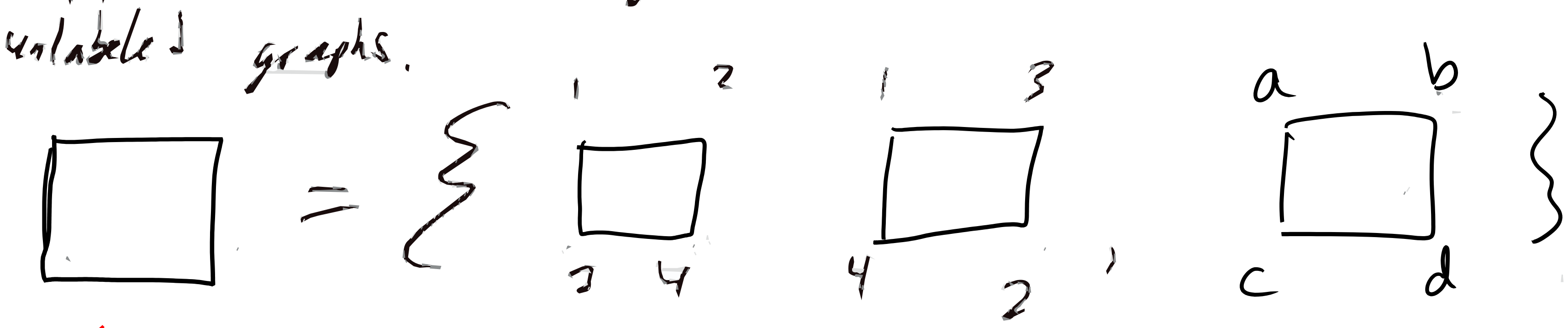
1) Symmetry $(x,y) \in R \Rightarrow (y,x) \in R$
2) Reflexivity $(x,x) \in R$
3) Transitivity $(x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$

Pf.
① Reflexivity
② Symmetry
③ Transitivity

Note. Equivalence Relations & Equivalence Classes

Def. An isomorphism class of graphs is an equivalence class of graphs under isomorphism

We often denote such equivalence classes by writing unlabeled graphs.



Unlabeled drawings refer to isomorphism classes of graphs.

Definition:

The *unlabeled path* and *unlabeled cycle* with n vertices are denoted P_n and C_n, respectively. C_n is sometimes also called an n-cycle

Definition:

A *complete graph* on n vertices is a simple graph with all pairs of distinct vertices adjacent, and is denoted K_n

Question:

How many edges are there in a complete graph with n vertices?

$K_{r,s}$ = biclique or complete bipartite graph.

When we name P_n, C_n, K_n etc we name the isomorphism class of a graph.

Question:

How many edges are there in $K_{r,s}$?

$rs \leftarrow$ what if you construct a complete graph by breaking it up into a bipartite graph first.

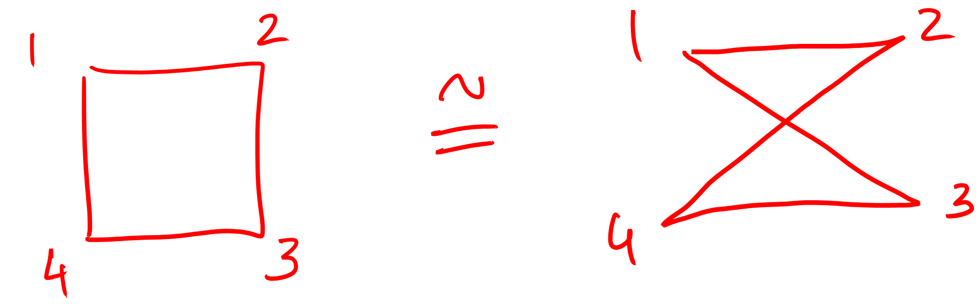
$rs + \binom{r}{2} + \binom{s}{2} = \binom{r+s}{2}$

Counting graphs

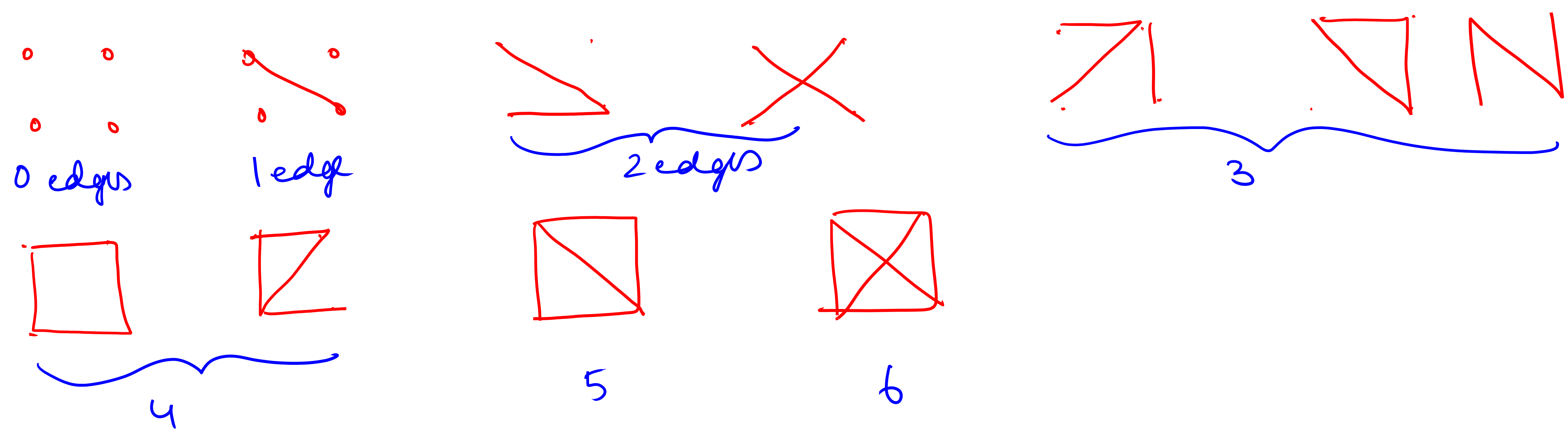
- How many graphs on $\{1\ 2\ 3\ 4\}$? K_4 has $\binom{4}{2} = 6$ edges.

Each edge can be chosen to be in the graph or not $\Rightarrow 2^6$ graphs $= 64$

BUT



There are only 11 isomorphism classes!



How many members does each isomorphism class contain.

Proposition:
If two simple graphs H and G are isomorphic, then their complements are also isomorphic.

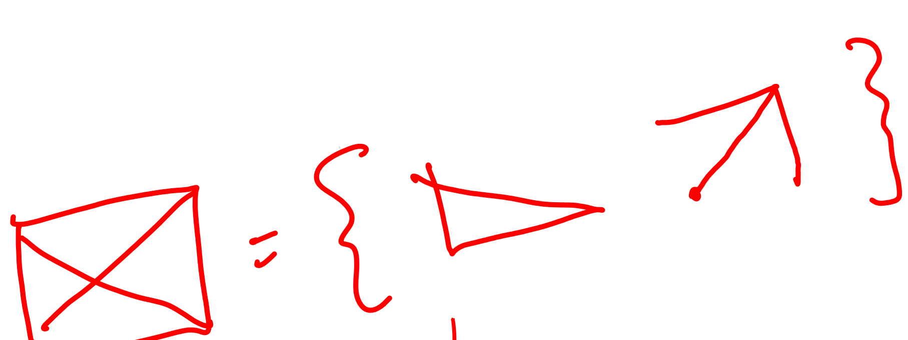
A complement of G is a graph $\bar{G}$ with the same vertices as G, and $uv \in E(\bar{G})$ iff $uv \notin E(G)$

Decomposition of Graphs and Some Special Graphs

Question:
Consider the graph P_4. What is its complement?"

$P_4 \cong \bar{P}_4$

Definition:
We say a graph is *self-complementary* if it is isomorphic to its complement.



Definition:
A *decomposition* of a graph G is a list of subgraphs of G such that each edge in $E(G)$ appears in exactly one subgraph in the list.

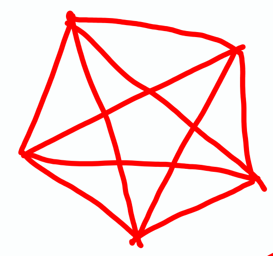
Proposition:

A graph H with n vertices is self-complementary if and only if K_n has a decomposition consisting of two copies of H.

I like this example

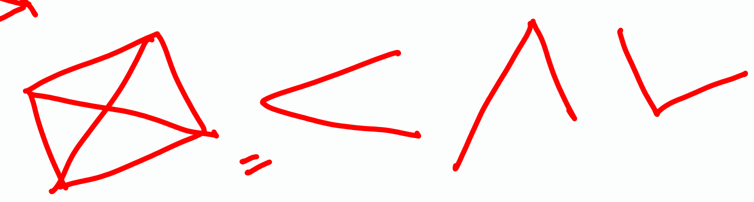
Example:

K_5 is two copies of C_5



Example:

K_4 is three copies of P_3 →



Pf:

If
$$\begin{tikzcd} & K_n \arrow[dl] \arrow[r] & G_2 \arrow[dl, "f_2"] \\ G_1 \arrow[d, "f_1"'] & & \\ H & & \end{tikzcd}$$

G_1 and G_2 must have same # of vertices & edges $\Rightarrow$ G_1 & G_2 must have n vertices.

Assume G_1 & G_2 have the same vertex set then. If $uv \in G_1$, then we must have $uv \notin G_2$, and vice versa. Therefore $G_2 = \overline{G_1}$

If H is self complementary, make the same arrangement as above. Each edge $uv \in K_n$ must be in H or $\overline{H}$.

Note:

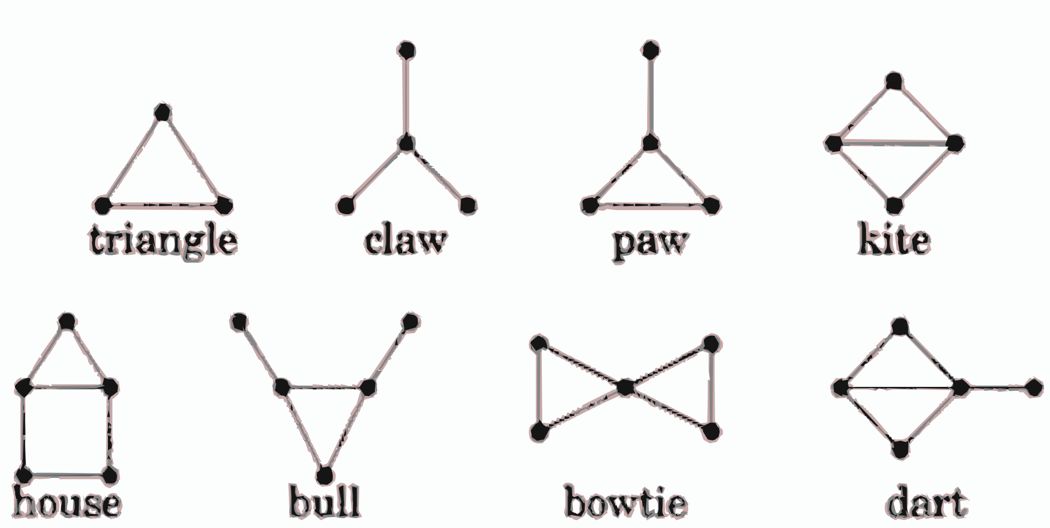
In many computational applications, we decompose complicated shapes into *triangulations*. Doing this is fundamentally about graph decompositions!

↳ really? → To ignore for now.

Note:

Lots of graphs have cute names, some of which are commonly used and others less so

Constellation graphs? another project.

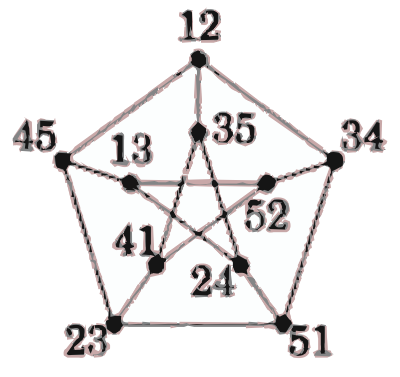


Which of these are self complementary?

There are myriad other specific graphs of interest.

Definition:

The *Petersen graph* is the simple graph G with $V(G)$ the 2-element subsets of a 5-element set with edges joining each pair of disjoint subsets. ✓



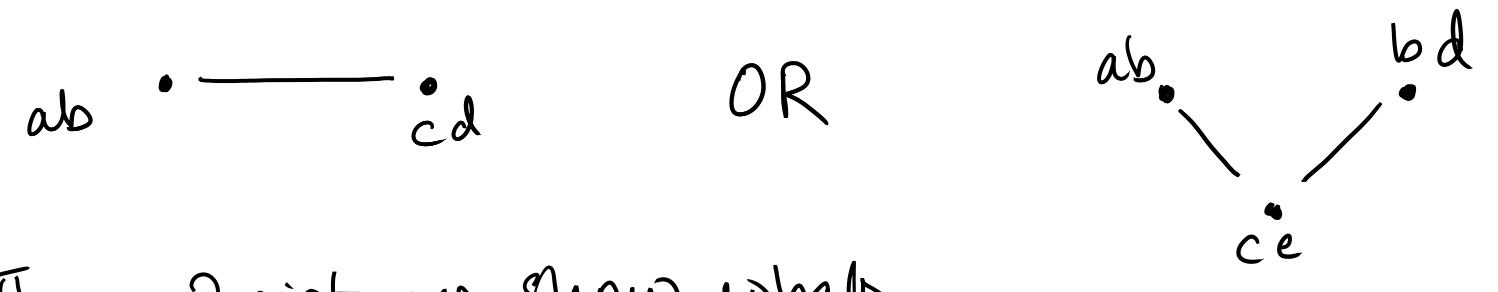
$\{1\,2\,3\,4\,5\} \rightsquigarrow \binom{5}{2} = \frac{5\cdot 4}{2} = 10$

$12 \Leftrightarrow 34$ | $12 \ldots 23$, 45

Can ignore & argue directly

Proposition: (for the class)

Given any two distinct points in the Petersen graph, there exists a unique path of length either 1 or 2 (but not both) connecting them.



These 2 pictures show what's going on.

Definition:

The *girth* of a graph is the length of the shortest cycle contained in the graph. If the graph contains no cycle, the girth is said to be infinite.

*Ex

Claim:

Any complete graph with at least three vertices has girth 3.

Any complete bipartite graph with at least 2 vertices in each partite set has girth 4.

Claim:

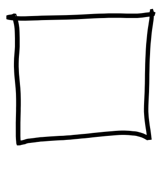
The Petersen graph has girth 5.

Pf: C_1 C_2 C_3 (simple cycles start at $n = 3$)

Is C_3 possible $\{ab, cd\} \leftarrow$ adjacent since they don't share a vertex.

xy has to be adjacent to both ab and cd. $\Rightarrow \{x, y\} \cap \{a, b, c, d\} = \phi$

But this is not possible since only 5 labels are available.

C_4?  $\rightarrow$ This has two paths of length 2 between non adjacent vertices. Not possible (see prev.)

Note: Every permutation of $\{1, 2, 3, 4, 5\}$ preserves disjointness: $\{a, b\} \cap \{c, d\} = \phi$

$\Leftrightarrow \{6a, 6b\} \cap \{6c, 6d\} = \phi \Rightarrow$ Petersen has $5!$ isomorphisms.

Consider exercise 43.

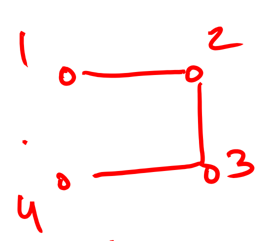
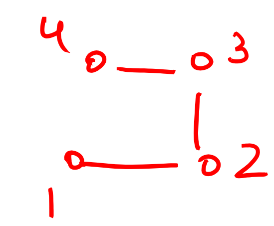
Definition:

An *automorphism* of a graph G is an isomorphism from G to G.

$(V, R \subset V \times V)$ $(V, R \subset V \times V)$

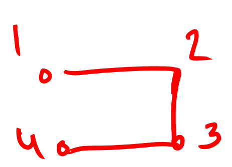
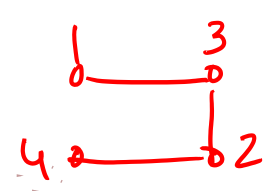
(MUST preserve BOTH vertex & edge rels)

Example:


automorphic to


$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

2 automorphisms.

BUT

$\cong$


Definition:

A graph G is *vertex-transitive* if, for every pair $u, v \in V(G)$, there exists an automorphism of G that maps u to v.

An easy example is the 4-cycle.

P_4 is not vertex-transitive. (since there are only 2 automorphisms)

Claim:

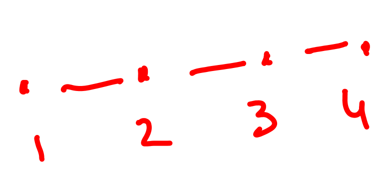
The Petersen graph is vertex-transitive.

Note:

Suppose a graph G is vertex-transitive. If we prove a property of the graph is true for some specific vertex, it must be true for all vertices!

worth doing this example.

List of all possible automorphisms = S_4 (permutations)



σ is a transposition

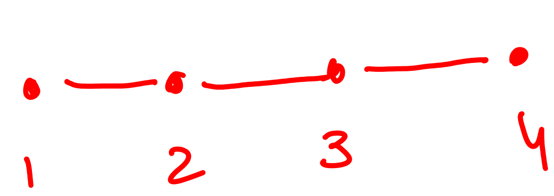
$\sigma(1)=3$
$\sigma(2)=2$
$\sigma(3)=1$
$\sigma(4)=4$

Is it an automorphism?

$E(P_4) = \{12, 23, 34\}$ $\quad E(\sigma P_4) = \{12, 23, 14\}$ ✗

and so on.

$\sigma = (4\ 3\ 2\ 1)$



These are the only 2 automorphisms!

Isomorphism and Automorphism

A graph is a triple $G = (V, E, R)$

$R : E \to V \times V / \sim$ is a map. Allows for multiple edges and loops.

$\uparrow$ Undirected case

In the simple case, we have

$E \subset \{ B \subset V : |B| = 2 \}$ a subcollection of the cardinality two subsets of V.

Given G, H an isomorphism is a (pair of bijections)

$f : V(G) \longrightarrow V(H)$

$f : E(G) \longrightarrow E(H)$

In the simple case enough to specify $f : V(G) \longrightarrow V(H)$ and

$uv \in E(G) \iff f(u) f(v) \in V(H)$. $\star 1$

Take $G = (\{1, 3, 5\}, \{13, 35\})$

$H = \{ \{pig, cat, tree\}, \{pig\ cat, cat\ tree\})$

Clearly G and H are isomorphic. So basically isomorphism is a relabelling of the vertices of G.

There are ∞ ly many isomorphisms.

BUT, if one fixes the vertex labels, we can count isomorphisms, like we did for graphs on 4 vertices labeled $\{1,2,3,4\}$. There were 64 graphs, but 11 isomorphism classes.

<u>Automorphism</u> $f: G \longrightarrow G$ bijection. Here

$f: V(G) \longrightarrow V(G)$ in the simple case.

The # of candidate automorphisms are finite! This is because if $V(G) = \{1, \ldots, n\}$ the canonical labels. Any bijection on $V(G)$ is a permutation on $[n]$!